On soliton structure of the Vakhnenko equation with a 'dissipative' term: a peculiar fission phenomenon

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Abstract

In the present work, we investigate the soliton structure of the Vakhnenko equation with a 'dissipative' term, by means of the Hirota's method. As a result, we unearth three kinds of soliton solutions depending upon the dissipation parameter. We further find that the scattering behavior among such structures exhibits a peculiar feature that may be the fission phenomenon.

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I. INTRODUCTION

Nonlinear systems and their related waves in different branches of physics have recently been investigated in detail through theory and experiments. Since the investigation of the nonlinear transmission line, pioneered by Hirota and Sazuki [1], various, kinds of nonlinear wave propagation have been discussed [2, 3, 4, 5]. Integrable models of such systems have been applied to many fields of research such as condensed matter [6, 7], fluid mechanics [8], plasma physics [9], optics [10, 11], communications [12, 13], chemistry [14], and biology [15, 16]. These integrable models which may take the form of nonlinear partial differential equations, may possess some kinds of self-confined solitary waves solutions known as solitons.

Since Gardner, Greene, Kruskal and Miura have solved the Korteweg-de Vries equation using the inverse scattering method [17], the modern theory of solitons has been deeply studied in mathematics, and widely applied in physics. A wealth of interesting properties of soliton equations have been found . Indeed, a soliton equation may possess B$\ddot{a}$cklund and Darboux transformations, the Lax-pair, N-soliton solutions, a bilinear form, a zero-curvature form, the Painlev$\acute{e}$ property, infinitely many symmetries and conservation laws, just to name a few [18].

Recently, Vakhnenko [19] have investigated high-frequency perturbations in a relaxing barothropic medium. He has then derived a new nonlinear evolution equation known as Vakhnenko equation [19] given by

$$u_{tx} + (uu_x)_x + \alpha u_x + u = 0, \tag{1}$$

where u may stand for a physical observable, and subscripts t and x appended to u may refer to partial differentiation with respect to these independent variables. It may be noted here that the αu_x-term may be regarded and defined as the 'dissipation' term [19]. Without 'dissipation', this equation which has been subject to many investigations [20, 21, 22, 23, 24], may possess loop-like soliton solutions. As far as we are concerned, few investigations [19] of Eq. (1) may have been done. Vakhnenko [19] has pointed out that Eq. (1) may possess ambiguous solutions in the form of a solitary wave, and he has proved that the dissipative term, with a dissipative parameter α less than that limit value, does not destroy the loop-like solutions. One underlying query that may obviously be asked may be related to the existence of other kinds of solutions when the dissipation parameter is greater or equal to this limit value. Besides, the question of soliton structure of such solutions may be investigated.

In order to provide some answers to these queries, the study of the fundamental role played by the dissipation parameter may be done around the following points. In Sec. II, the different kinds of the one-soliton solution are presented. In Sec. III, the two-soliton solutions and their scattering behavior are depicted. Some peculiar phenomenon such as the fission phenomenon, has been identified. Sec. IV is devoted to a brief summary.

II. BILINEARIZATION OF THE VAKHNENKO EQUATION AND CONSTRUCTION OF THE ONE-SOLITON SOLUTIONS

Introducing new independent variables X and T as follows [22]

$$x = T + \int_{-\infty}^{X} U(s,T)ds + x_0, \quad t = X, \tag{2}$$

where $u(t,x) \equiv U(X,T)$, and x_0 stands for an arbitrary constant, setting $U = W_X$, Eq. (1) transformed to [22]

$$W_{XXT} + W_X W_T + \alpha W_T + W_X = 0. \tag{3}$$

By taking

$$W = 6\left(\ln F\right)_X, \tag{4}$$

Eq. (3) may take the following coupled bilinearized forms

$$\left(D_T D_X^3 + D_X^2\right) F \cdot F + \alpha G F = 0, \tag{5a}$$

$$D_T\left(D_X^2 F \cdot F\right) \cdot F^2 - G F^3 = 0, \tag{5b}$$

where G may stand for an arbitrary function. The quantities D_T and D_X denote the Hirota operators. According to the usual procedure, the soliton solutions may be constructed by expanding F and G in suitable formal power series.

Thus, the one-soliton solutions may be given by

$$F = 1 + \exp(2\eta), \tag{6}$$

where

$$\eta = KX - \omega T + \eta_0, \tag{7}$$

quantities K and ω standing for wave number and angular frequency, respectively. The dispersion equation may be given by

$$2\left(\alpha+2K\right)\omega-1=0, \tag{8}$$

and $\omega = Kv$, $v > 0$ being the velocity of the wave. Combining Eqs. (4) and (6) may lead to

$$W=6K\left[1+\tanh\left(\eta\right)\right], \tag{9}$$

and hence

$$U=U_M\texttt{sech}^2\left(\eta\right), \tag{10}$$

where the amplitude $U_M = 6K^2$.

In order to discuss the different types of solutions, it seems worthy to consider the following equation

$$\partial_T=\left[1-6K\omega\texttt{sech}^2\left(\eta\right)\right]\partial_x. \tag{11}$$

Thus, setting $\lambda = 6K\omega$, we may distinguish three kinds of solutions according to the cases $\lambda < 1$, $\lambda = 1$, and $\lambda > 1$. As a result, we find that

1. for $\alpha < \frac{1}{\sqrt{6v}}$, loop-like solution may be obtained (see FIG. 1 where loop is represented by the solid line);

2. for $\alpha = \frac{1}{\sqrt{6v}}$, cusp-like solution may be obtained (see FIG. 1 where cusp is represented by the broken line);

3. finally, for $\alpha > \frac{1}{\sqrt{6v}}$, hump-like solution may be obtained (see FIG. 1 where hump is represented by the dotted line).

In FIG. 1, the previous kinds of soliton solutions have been depicted as variations of U/U_M vs x. It seems also worth depicting the domain in which these solutions may exist. In this view, we depict at FIG. 2, the variations of α vs the velocity of waves v. In this figure, the white area may refer to hump-like soliton solutions. It seems also worthy to compare the amplitudes of these solutions. The variations of U_M vs the velocity of waves v are depicted in FIG. 3 where the white area may refer to the loop-like soliton solutions. Thus, for a given velocity, the greatest amplitude may be due to loop-like solutions and the smallest amplitude belongs to hump-like solutions.

III. THE TWO-SOLITON SOLUTIONS

The solution to Eq. (5) corresponding to two-soliton solutions may be given by

$$F = 1 + \exp(\eta_2) + \exp(2\eta_2) + A\exp(4\eta_1) + B\exp(4\eta_2) + C\exp 2(\eta_1 + \eta_2), \tag{12}$$

where the phases η_i $(i = 1, 2)$ may be given by

$$\eta_i = K_i X - \omega_i T + \eta_{i0}, \quad (i = 1, 2), \tag{13}$$

and the dispersion relations

$$2\left(\alpha + 2K_i\right)\omega_i - 1 = 0, \quad (i = 1, 2) \tag{14}$$

where $\omega_i = K_i v_i$ $(i = 1, 2)$, v_i standing for velocities of waves.

The cœfficients A, B and C may be given by

$$A = \frac{\alpha}{2\left(\alpha + 6K_1\right)}, \tag{15a}$$

$$B = \frac{\alpha}{2\left(\alpha + 6K_2\right)}, \tag{15b}$$

$$C = \frac{2\alpha\left[\left(\omega_1 - \omega_2\right)\left(K_1^2 - K_2^2\right) + 2K_1K_2\left(\omega_1 + \omega_2\right)\right] + 4\left(\omega_2 - \omega_1\right)\left(K_1 - K_2\right)^3 + \left(K_1 - K_2\right)^2}{\left(K_1 + K_2\right)^2\left[2\left(\omega_1 + \omega_2\right)\left(\alpha + 2K_1 + K_2\right) - 1\right]}. \tag{15c}$$

In order to investigate the scattering behavior among these two-soliton solutions, we may firstly consider some underlying snapshots.

- For $\alpha = 1.2$, according to Eq. (11), two snapshots depicting loop-like and hump-like solitons may be depicted at FIGS. 4 and 5 at times $t = -15$ and $t = 11$, respectively;

- for $\alpha = 0.1$, the same phenomenon may be observed but we may see that the loop-like soliton splits into two other loops of the same amplitudes. This case may be depicted at FIGS. 6 and 7 at times $t = -15$ and $t = 11$, respectively. Thus, as a result, there may exist a value at which the loop-like soliton begins to split. It may be suggested that for velocities $v_1 = 0.24$ and $v_2 = 0.12$, there may not be splitting provided the relation $1.2 < \alpha < 2.6$ holds, whereas for $\alpha < 1.2$, splitting may be observed;

- for $\alpha = 2.6$ with velocities $v_1 = 0.24$ and $v_2 = 0.12$, according to Eq. (11), two snapshots depicting cusp-like and hump-like solitons may be depicted at FIGS. 8 and 9 at times $t = -15$ and $t = 11$, respectively;

- finally for $\alpha = 5.0$ with velocities $v_1 = 0.24$ and $v_2 = 0.12$, according to Eq. (11), two snapshots depicting large hump-like and small hump-like solitons may be depicted at FIGS. 10 and 11 at times $t = -15$ and $t = 11$, respectively.

Now, the full scattering behavior among these soliton solutions may be properly analyzed. As it may be observed, for negative values of time t, only one-soliton solution may be observed. For positive values of time t, the initial one-soliton solution splits into two other solitons with an increase in the amplitude. The soliton with the small amplitude may have hump-like shape. This kind of phenomenon that may refer to a fission, has recently been investigated by Morrison and Parkes [25] while deriving the N-soliton solution to a generalized Vakhnenko equation.

IV. SUMMARY

In this paper, we have investigated the soliton structure of the Vakhnenko equation [22] with a 'dissipative' term, by means of the Hirota's method. This model equation derived by Vakhnenko [22] may be underlying in description of high-frequency perturbations in a relaxing barothropic medium. As a result, we have unearthed three kinds of soliton solutions depending upon the dissipation parameter. These soliton solutions may have loop-like, cusp-like and hump-like shapes, respectively. We have further found that the scattering behavior among such structures exhibits a peculiar feature known as fission. Indeed, the initial one-soliton splits into two other kinds of solitons as time elapses from negative values to positive values. This kind of phenomenon may have been pointed out by Morrison and Parkes [25] while investigating the N-soliton solution to a generalized Vakhnenko equation. In order to provide the full detail on the interactions such as asymptotic bahavior of the scattering among the soliton solutions to the Vakhnenko equation [22] (see Eq. (1)), the same procedure described in Ref. [25] may be followed. Nonetheless, for some convenience due to the length of the paper, we do not report such result here. We may have only focused our interest to the effect of the dissipative parameter α on the Vakhnenko equation [22] (see Eq. (1)).

For a further interest, it may be worthy to investigate the N-soliton solutions ($N \geq 3$) to the Vakhnenko equation [22] (see Eq. (1)). Another interesting study may be to fix the dissipation parameter α, and subsequently investigate the effect of the velocities of waves on the system. Besides, following a recent work of Konno and Kakuhata [26] on rotating soliton

solutions to a coupled dispersionless system, it may be worth considering complex-valued angular frequency ω and wave number K into the dispersion equation. This may hopefully help investigating rotating soliton solutions to the Vakhnenko equation [22] (see Eq. (1)).

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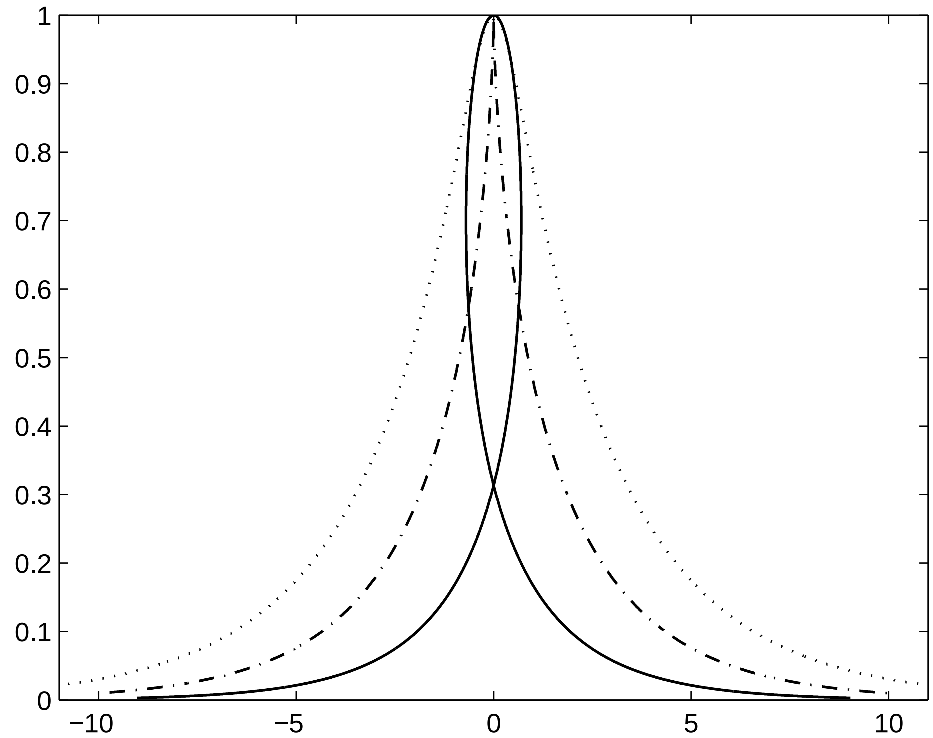

FIG. 1: One-soliton solutions U/U_M vs x: the solid, broken and dotted lines may represent the loop-like, cusp-like and hump-like soliton solutions, respectively.

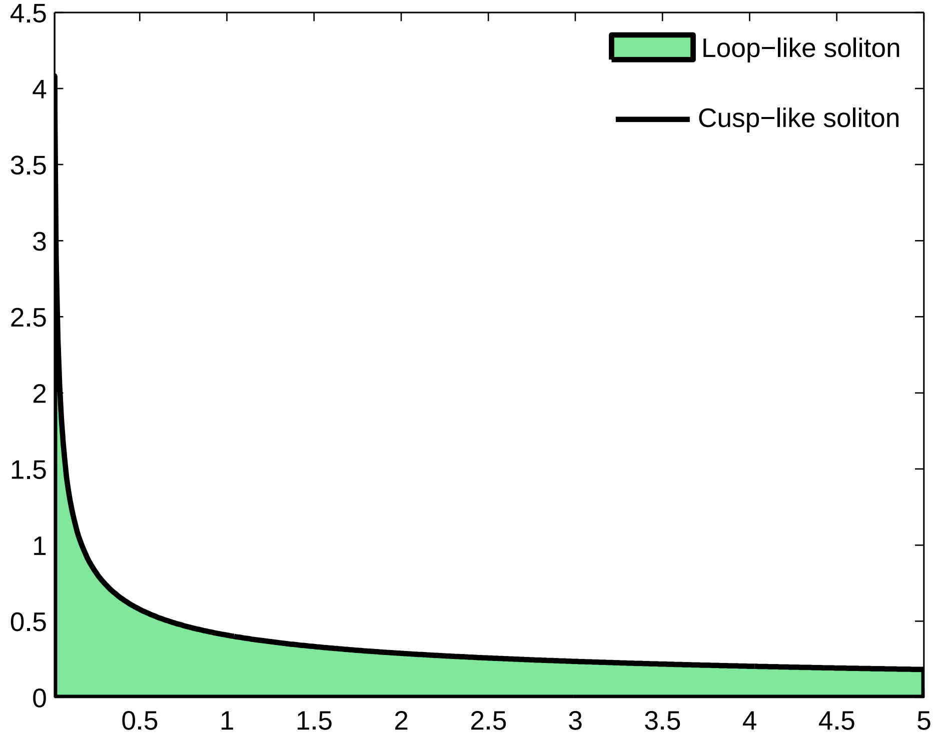

FIG. 2: Variations of the dissipation parameter α vs the velocity v of waves. White zone may refer to hump-like soliton solutions.

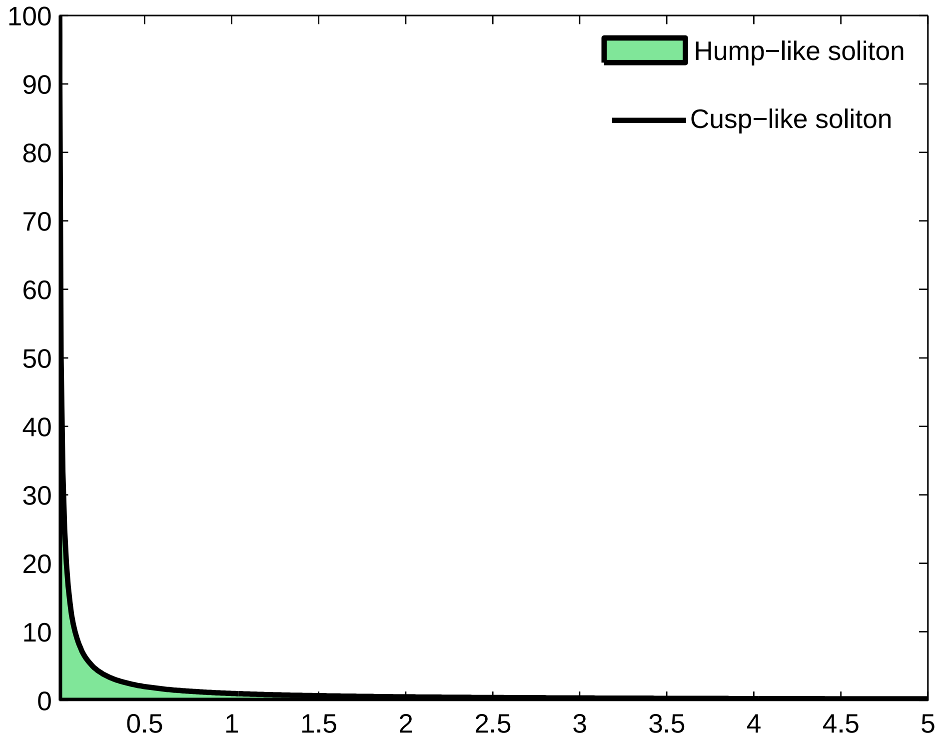

FIG. 3: Variations of the amplitude U_M of waves vs the velocity v of waves. White area may refer to loop-like soliton solutions.

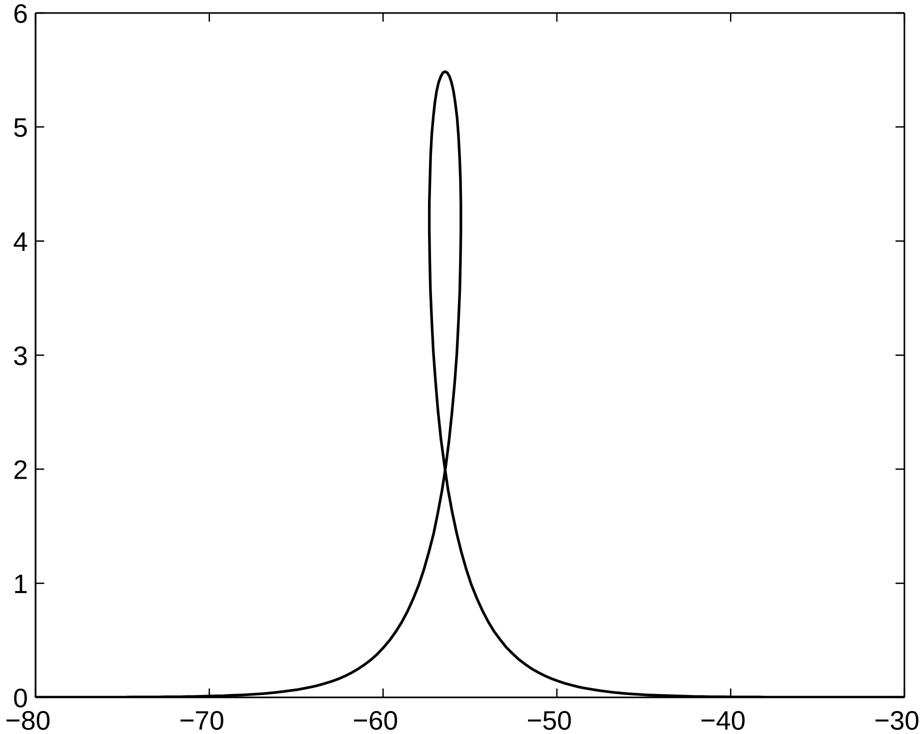

FIG. 4: Snapshot of loop-like soliton solution at time $t = -15$ from Eq. (12) for $\alpha = 1.2$, $v_1 = 0.24$ and $v_2 = 0.12$.

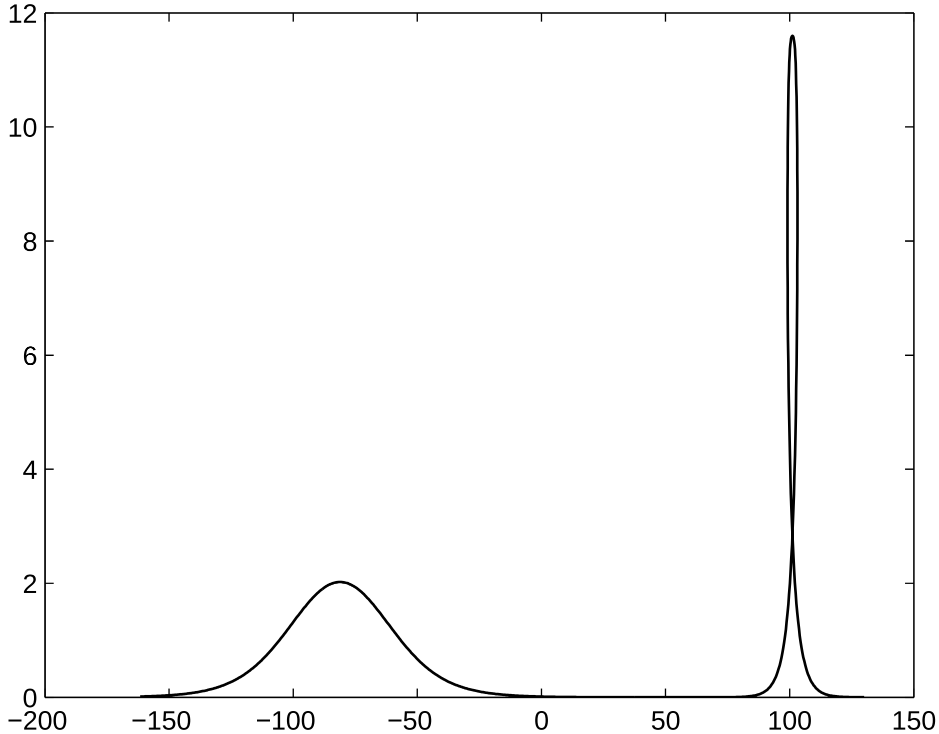

FIG. 5: Snapshot of loop-like soliton solution at time $t = 11$ from Eq. (12) for $\alpha = 1.2$, $v_1 = 0.24$ and $v_2 = 0.12$.

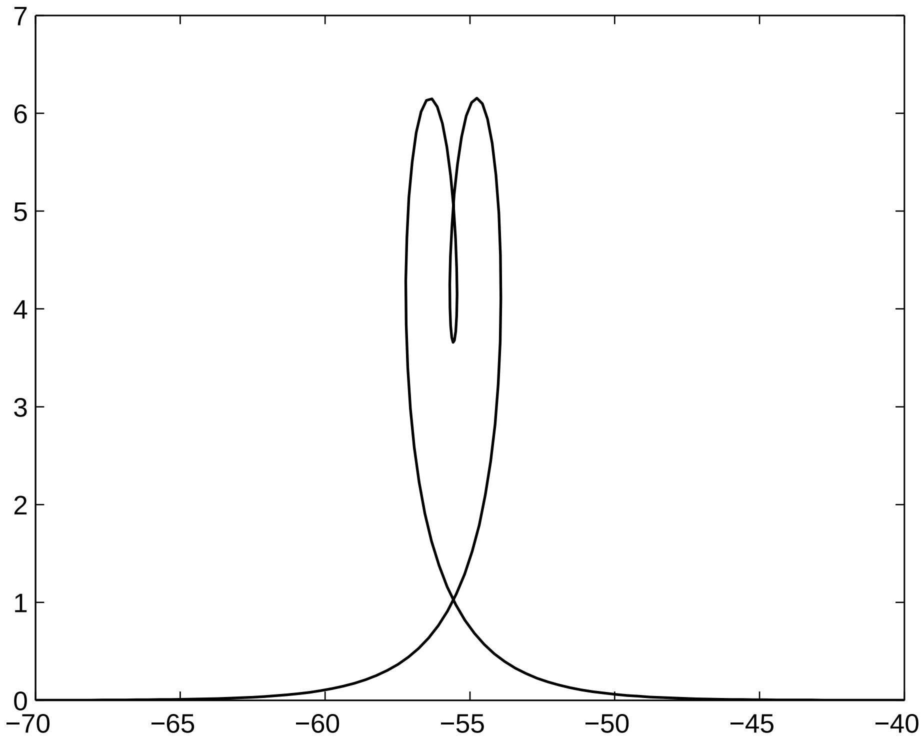

FIG. 6: Snapshot of splitting loop-like soliton solution at time $t = -15$ from Eq. (12) for $\alpha = 0.1$, $v_1 = 0.24$ and $v_2 = 0.12$.

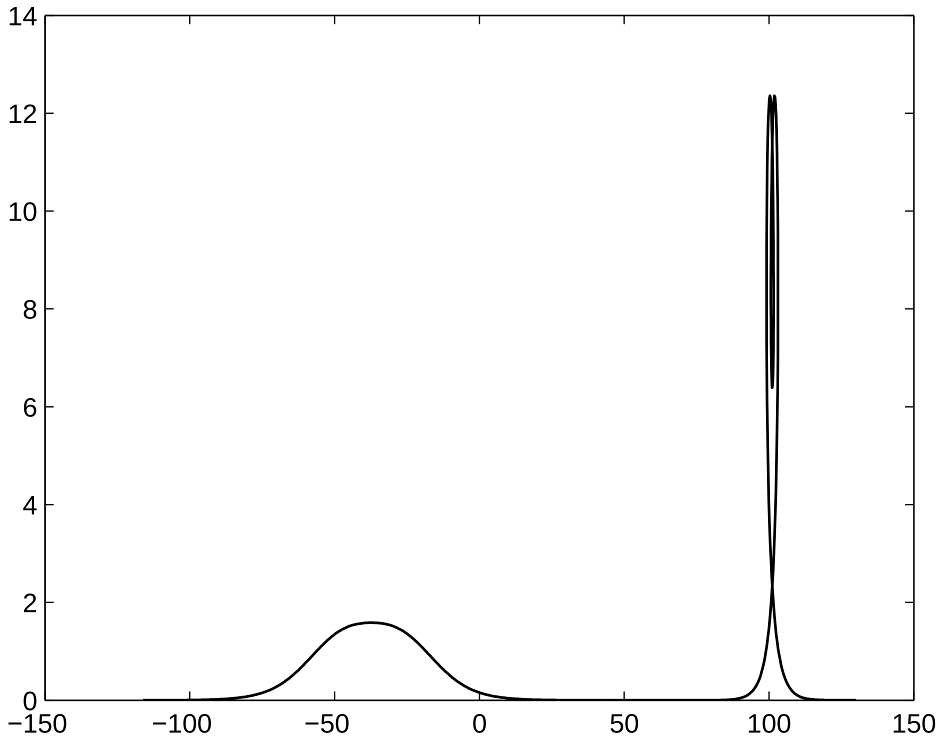

FIG. 7: Snapshot of splitting loop-like soliton solution at time $t = 11$ from Eq. (12) for $\alpha = 0.1$, $v_1 = 0.24$ and $v_2 = 0.12$.

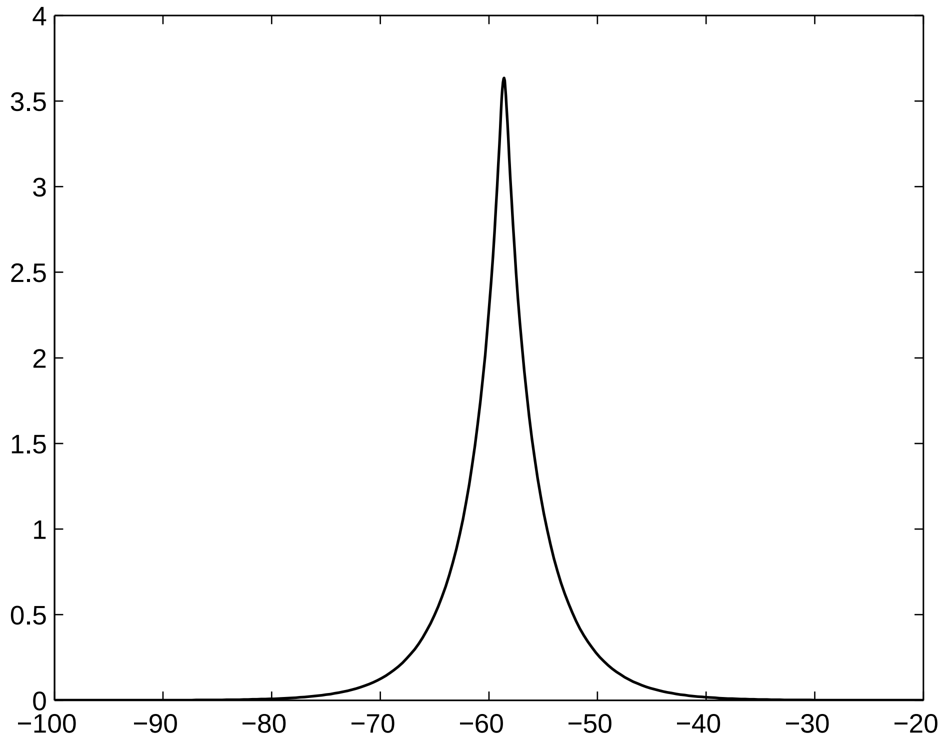

FIG. 8: Snapshot of cusp-like soliton solution at time $t = -15$ from Eq. (12) for $\alpha = 2.6$, $v_1 = 0.24$ and $v_2 = 0.12$.

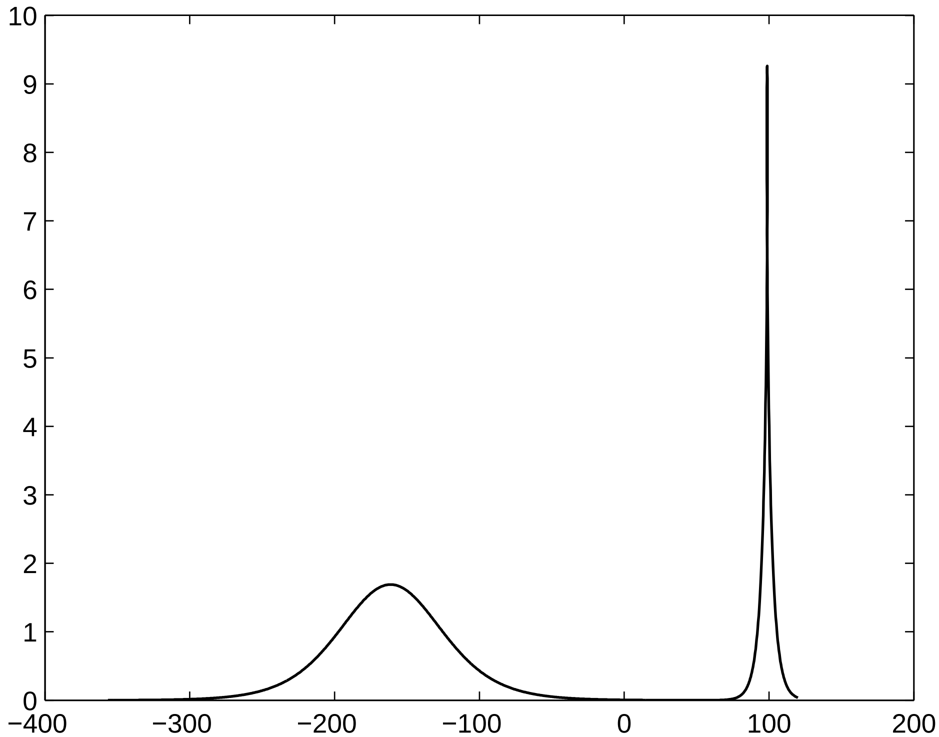

FIG. 9: Snapshot of cusp-like soliton solution at time $t = 11$ from Eq. (12) for $\alpha = 2.6$, $v_1 = 0.24$ and $v_2 = 0.12$.

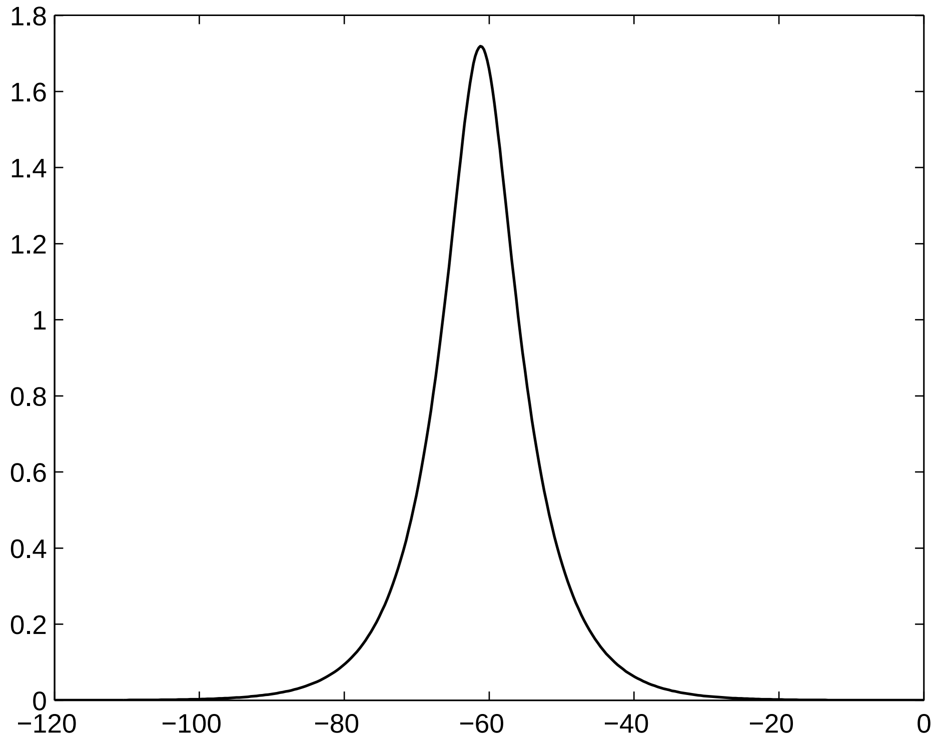

FIG. 10: Snapshot of hump-like soliton solution at time $t = -15$ from Eq. (12) for $\alpha = 5.0$, $v_1 = 0.24$ and $v_2 = 0.12$.

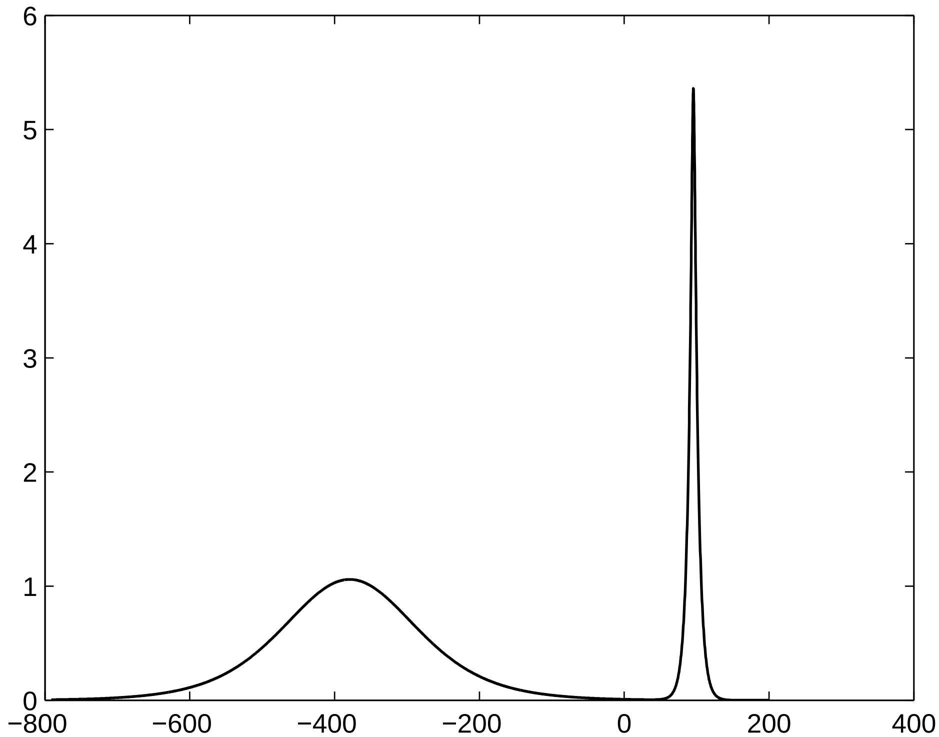

FIG. 11: Snapshot of hump-like soliton solution at time $t = 11$ from Eq. (12) for $\alpha = 5.0$, $v_1 = 0.24$ and $v_2 = 0.12$.